Variational approach to the problem of dark-soliton generation

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The creation of dark solitons from an arbitrary initial pulse in the system, described by the nonlinear Schrödinger equation, is considered by applying the variational method to the corresponding linear spectral problem. The initial pulse is a potential in the linear operator of the Zakharov-Shabat eigenvalue problem and the discrete spectrum of the problem determines the number and parameters of emerged solitons. The procedure for calculation of approximate values of the lowest- and higher-order discrete eigenvalues from spectral data of known (trial) potential is proposed. The application of this procedure to some examples shows qualitative agreement between variational and exact results. [S1063-651X(97)09409-9]

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I. INTRODUCTION

The problem of propagation of dark solitons in various media has remained the center of intensive experimental and theoretical investigations during past two decades. As is well known, dark solitons are localized excitations on a uniform background. In a problem of evolution of optical pulses in fibers, dark solitons represent troughs of intensity that are localized in time. The dynamics of optical solitons is governed by the nonlinear Schrödinger (NLS) equation [1–3]

$$iQ_t-(\sigma/2)Q_{xx}+|Q|^2Q=0. \qquad (1.1)$$

In this case $Q(x,t)$ is the slowly varying pulse envelope, t is the distance along a fiber, and x is the time in a frame of reference moving with the group velocity. Indices denote partial differentiation with respect to appropriate variables. Equation (1.1) describes two different classes of solitary waves depending on the sign of $\sigma=\pm1$. In this paper, we investigate the case $\sigma=1$ (dark solitons), which corresponds to the normal group-velocity dispersion. The propagation of nonlinear optical pulses in the region of anomalous group-velocity dispersion is also governed by Eq. (1.1), with $\sigma=-1$ (bright solitons).

The NLS equation arises in many areas of physics such as nonlinear optics and plasma physics. For instance, the diffraction of plane electromagnetic waves in a nonlinear defocusing medium and the propagation of perturbations in a Bose gas with repulsion are also described by the NLS equation [1–3].

As was discovered by Zakharov and Shabat [1], Eq. (1.1) is exactly integrable by means of the inverse scattering transform (IST) technique and it has an exact soliton solution in the form

$$Q_S(x,t)=[\lambda-i\nu\tanh(z)]\exp(i|q_0|^2t+i\phi), \qquad (1.2)$$

where $z=\nu(x+\lambda t)$, $\nu^2=|q_0|^2-\lambda^2$, $\tan(\phi)=\nu/\lambda$, $|q_0|$ is the background wave amplitude, and ν and λ are the amplitude and velocity of the dark soliton, respectively. Obviously, it is very complicated to create the dark solitons in a shape described by Eq. (1.2) exactly. So it seems interesting to consider the problem of generation of dark solitons by an arbitrary initial pulse. The fact that Eq. (1.1) is exactly integrable by the IST method carries a guarantee that any initial pulse will evolve into a set of dark solitons and radiation. So the main attention in our consideration is devoted to the determination of the number of emerging solitons and their parameters. As is well known, one should solve the corresponding Zakharov-Shabat eigenvalue problem in order to obtain the information about emerging solitons. There are few initial pulses for which the Zakharov-Shabat problem can be solved exactly; among them are ''black'' [1] and ''gray'' [4] boxes (see below) and a tanh pulse [5].

In the present paper we aim to use a variational approach for obtaining approximate eigenvalues, which determine the parameters of solitons. Such an approach, widely used in the quantum theory, was applied in [6–8] to bright-soliton generation. In these papers the explicit formulas for eigenvalues were obtained for different kinds of complex potentials. Reference [7] suggests an interesting idea for taking all eigenvalues based on the eigenfunctions corresponding to zero eigenvalue. Here we should note that the problem considered in the present paper has specific properties, unlike the problem corresponding to the ''bright'' NLS case, and has been studied by other authors. The main differences are (i) the linear operator is the Hermitian operator, so its eigenvalues are real and (ii) the scattering potential [initial condition for Eq. (1.1)] has nonzero values at infinity. As a result, the above-enumerated differences cause the specific behavior of the discrete spectrum (see [1,9]).

The paper is organized as follows. Section II contains the basic idea of the analytical method, which gives an approximate solution of the eigenvalue problem. We propose a simple tool based on the variational method for finding both the lowest- and higher-order eigenvalues. In our consideration the discrete spectrum corresponding to the given potential may be obtained from eigenfunctions and eigenvalues of a trial potential for which the problem can be solved exactly. The best approximation may be achieved by varying the parameters of the trial potential. In order to avoid tedious calculations we suggest also a simple rule for the estimation of optimal values of variational parameters. The rule is based on the use of integral invariants of the NLS equation. Certainly this step is beyond the variational scheme, but it simplifies calculations essentially. In Sec. III we consider the application of the method to different initial pulses. As our

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main purpose is to demonstrate the possibility of the variational approach, for simplicity we consider only real initial pulses $Q(x,0)$. We find good agreement between approximate and exact solutions. Section IV summarizes the results of the paper. In the Appendix, eigenfunctions and spectral parameters of the piecewise constant potential, which may be used as the trial potential, are given.

II. VARIATIONAL APPROACH TO THE ZAKHAROV-SHABAT EIGENVALUE PROBLEM

Because of the integrability of Eq. (1.1) by the IST method, it follows that the evolution of the initial pulse $Q(x,0)=q(x)$, where $q(x)\rightarrow|q_0|e^{i\theta}$ for $x\rightarrow-\infty$ and $q(x)\rightarrow|q_0|$ for $x\rightarrow+\infty$ [Cauchy problem for Eq. (1.1)], is reduced to the spectral problem of the Zakharov-Shabat system [1]

$$F\Psi=\lambda\Psi,\quad F=\begin{pmatrix} i\partial/\partial x & -iq^*(x)\\ iq(x) & -i\partial/\partial x\end{pmatrix},\quad \Psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\tag{2.1}$$

with the boundary conditions

$$\begin{gathered}\Psi\rightarrow\begin{pmatrix}1\\ Ce^{i\theta}\end{pmatrix}\exp(-i\zeta x),\quad x\rightarrow-\infty\\ \Psi\rightarrow a(\lambda)\begin{pmatrix}1\\ C\end{pmatrix}\exp(-i\zeta x)+b(\lambda)\begin{pmatrix}-C\\ 1\end{pmatrix}\exp(i\zeta x),\\ x\rightarrow+\infty,\end{gathered}\tag{2.2}$$

where $C=i(\lambda-\zeta)/|q_0|$, $\zeta=[\lambda^2-|q_0|^2]^{1/2}$, $a(\lambda),b(\lambda)$ are scattering coefficients, and an asterisk stands for complex conjugation.

The discrete spectrum of the problem (2.1) and (2.2) can be found from the condition

$$a(\lambda)=0.\tag{2.3}$$

As shown in [1,9], the roots λ_n, $n=1,\ldots,N$, of the given equation lie in the real axis between $-|q_0|$ and $|q_0|$. The number of solitons emerging from the initial pulse coincides with the number N of discrete eigenvalues of the spectral problem (2.1). Parameters λ_n and $\nu_n=\zeta_n/i$ represent the parameters of the nth soliton [see Eq. (1.2)].

Although the system (2.1) is linear, its general solution for arbitrary initial pulse is not available. Therefore, every attempt to develop approximate analytical methods in order to obtain eigenvalues has an interest. For this purpose we reformulate the problem (2.1) and (2.2) as a variational problem. There are many ways to build a variational functional corresponding to the problem (2.1), but in this paper we put forth the reformulation based on the functional corresponding directly to spectral parameter λ (see, e.g., [10]):

$$\lambda=\langle\Psi^+F\Psi\rangle/\langle\Psi^+\Psi\rangle,\tag{2.4}$$

where angular brackets denote integration in x from $-\infty$ to $+\infty$ and the dagger corresponds to Hermitian conjugation. One can consider Eq. (2.4) in the following way. If we insert exact eigenfunctions, corresponding to the potential $q(x)$, into this functional, the right-hand side of Eq. (2.4) gives an exact eigenvalue. But substitution of any other (trial) functions $\Psi_{\rm tr}(x,\{c\})$ instead of exact solutions into this functional gives approximate eigenvalues of the problem. Here $\{c\}$ denotes the set of K parameters $c_1,c_2,\ldots,c_K$. The parameters corresponding to the best approximation of eigenvalues are determined from the condition

$$\delta\lambda(\Psi(x,\{c\}))=0\quad\text{or}\quad\partial(\Psi(x,\{c\}))/\partial c_k=0,\qquad k=1,\ldots,K.\tag{2.5}$$

Of course, the choice of good trial functions is a crucial step of the variational approach and requires deep insight into the physics of the problem in order to obtain more precise results.

This procedure gives approximate value of the lowest (nearest to 0) eigenvalue λ_1. To find higher eigenvalues λ_n, $n=2,\ldots,N$ one should take the functions that are orthogonal to the functions already used.

For convenience in choosing appropriate trial functions we propose to use the eigenfunctions of the (trial) potential $q^0(x,\{c\})$, for which analytical solution can be obtained exactly, i.e., the set of eigenvalues $\lambda_n^0(\{c\})$ and eigenfunctions $\Phi(x,\{c\})=(\varphi_1(x,\{c\}),\varphi_2(x,\{c\}))^T$. The advantages of the approach are as follows: (i) The trial functions automatically satisfy the necessary boundary (2.2) and other conditions and (ii) these functions are the set of orthogonal functions and therefore one may use them to obtain higher-order eigenvalues λ_n.

Making use of the functional (2.4) and Eq. (2.1), we have the basic formula for approximate eigenvalues λ, corresponding to the potential $q(x)$ from the spectral data $\lambda^0,\varphi_1,\varphi_2$ of the trial potential $q^0(x,\{c\})$:

$$\lambda=\lambda^0-2\,\frac{\langle\mathrm{Im}[(q-q^0)\varphi_1\varphi_2^*]\rangle}{\langle|\varphi_1|^2+|\varphi_2|^2\rangle}.\tag{2.6}$$

Note that in this equation the index n of $\lambda,\lambda^0,\varphi_1,\varphi_2$ is omitted, i.e., the equation is valid for the whole discrete spectrum.

The different approach based on the variation of the Lagrangian of the Zakharov-Shabat problem, corresponding to bright-soliton propagation, is considered in Refs. [6,7]. Explicit formulas for approximate eigenvalues and for conditions of eigenvalues number are obtained for different potentials. In the case of dark solitons the Lagrangian L of Eqs. (2.1) and (2.2) has the form

$$L=\tfrac{1}{2}\int_{-\infty}^{\infty}[\psi_2\psi_{1x}-\psi_1\psi_{2x}+2i\lambda\psi_1\psi_2+q\psi_1^2-q^*\psi_2^2]dx.\tag{2.7}$$

Due to specific properties of the spectral problem (2.1) and (2.2) it is impossible to use the procedure of Refs. [6,7] directly. However, following these works, one can take as the trial function the product of known eigenfunction $\Phi(x,\{c\})$ to parameter A: $\Psi=A\Phi(x,\{c\})$. In Ref. [7] the eigenfunction for $\lambda=0$, corresponding to the given potential, is chosen as the function $\Phi(x,\{c\})$; here we choose the

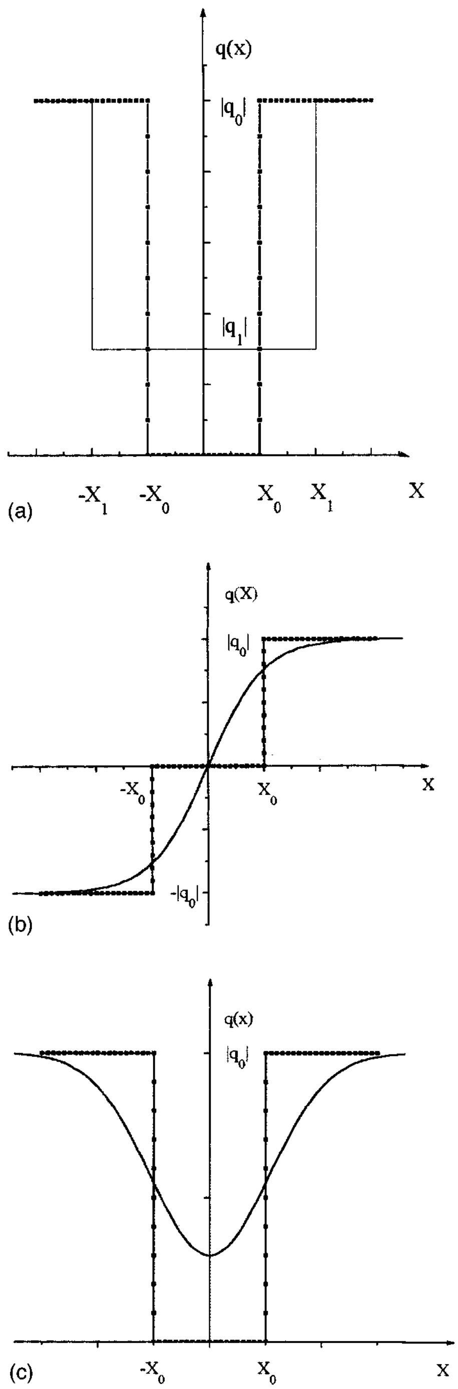

FIG. 1. Given (solid curve) and trial (dotted curve) potentials. (a) Gray and black boxes. (b) tanh pulse and antisymmetric black box. (c) Dark Gaussian pulse and black box.

eigenfunction of the trial potential $q^0(x,\{c\})$ as the function $\Phi(x,\{c\})$. Then we have from the condition $\partial L/\partial A=0$ the relation

$$\lambda=\lambda^0+\frac{i}{2}\frac{\langle[(q-q^0)\varphi_1^2-(q^*-q^{0*})\varphi_2^2]\rangle}{\langle\varphi_1\varphi_2\rangle}. \qquad (2.8)$$

Equation (2.8), for the estimation of eigenvalues, has free parameters $\{c\}$. The variation of L in $\{c\}$, taking into account Eq. (2.8), does not lead to new equations for parameters $\{c\}$. As a matter of fact, this difficulty was, e.g., in Ref. [6] also, where the final result has an arbitrary parameter. The problem appeared from the choice of the functional in the form (2.7). Therefore, in the following, we shall deal with the functional (2.4) only.

To determine the best approximation of λ it is necessary to vary Eq. (2.6) with respect to parameters $\{c\}$ [see Eq. (2.5)] and to insert the optimal values $\{c_{\rm opt}\}$ into Eq. (2.6). In many cases this procedure leads to complicated implicit equations for parameters $\{c\}$, so it is useful to have a relation for the estimation of them. On the other hand, it is known that Eq. (1.1) has an infinite set of integral invariants. For example, the first invariant (‘‘number of particles’’) has the form

$$I_1[q(x)]=\int_{-\infty}^{\infty}[|q_0|^2-|q(x)|^2]dx. \qquad (2.9)$$

If the trial functions depend only on one parameter c, then one can define it from the condition

$$I_1[q(x)]=I_1[q^0(x,c)]. \qquad (2.10)$$

We note that this step gives a simple and effective rule for obtaining $c_{\rm opt}$. In [11] integrals of motion were used to obtain the parameters of solitons. Here we use them for the determination of the variational parameters. Although the estimation (2.10) is rough, the approximate eigenvalues are close to the exact ones (see Sec. III).

Now we would like to note other possible applications of Eqs. (2.6) and (2.8). Let the potential q differ slightly from q^0, so that $\max[|q-q^0|]\sim\epsilon\ll 1$. Then, Eqs. (2.6) and (2.8) represent a result of perturbation theory to first order in ϵ. In this case we can investigate the influence of small (periodic and random) modulations of the initial pulse on the creation and propagation of dark solitons. Another application of Eqs. (2.6) and (2.8) is the possibility of taking the condition for emerging new eigenvalues (solitons). Let the initial condition also depend on the set of parameters $\{p\}$, i.e., $q(x,\{p\})$. Since by changing parameters $\{p\}$ the new eigenvalues ‘‘appear’’ from the point $\lambda=\pm|q_0|$, the threshold values of $\{p\}$ can be determined from the condition $|\lambda(\{p\})|=|q_0|$.

III. APPLICATION OF VARIATIONAL PROCEDURE

In this section we illustrate the possibility of the variational method by applying it to some particular examples. In further calculations we use Eq. (2.6) and the condition (2.10) for the evaluation of λ. It should be noted that Eq. (2.8) with condition (2.10) gives the same values of variational parameters as Eq. (2.6). It can be explained by the symmetry of eigenfunctions of the system (2.1) and (2.2). To obtain suitable trial functions we have solved the problem (2.1) for a piecewise constant potential

$$u(x)=\begin{cases}|q_0|\exp(i\theta_0) & \text{for } x<-d\\ |q_1|\exp(i\theta_1) & \text{for } |x|\leqslant d\\ |q_0| & \text{for } x>d.\end{cases} \qquad (3.1)$$

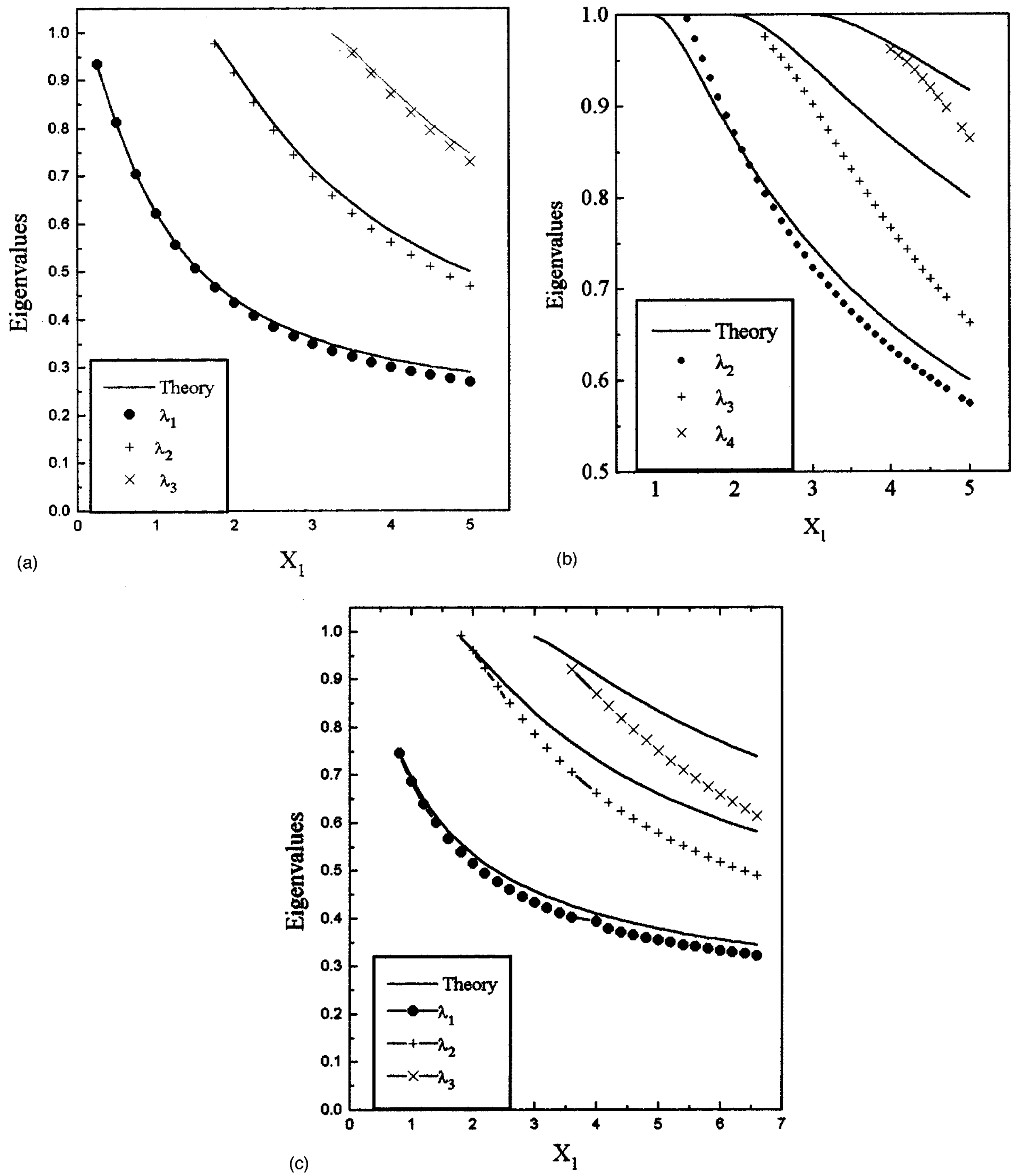

FIG. 2. Dependence of three eigenvalues on potential parameters. (a) Gray box: $\lambda_1,\lambda_2,\lambda_3$ as a function of x_1 for $q_1=0.2$. (b) tanh pulse: $\lambda_2,\lambda_3,\lambda_4$ ($\lambda_1=0$) as a function of x_1. (c) Dark Gaussian pulse: $\lambda_1,\lambda_2,\lambda_3$ as a function of x_1 for $\alpha=0.8$.

The expressions of spectral data $a(\lambda)$ and $b(\lambda)$ of eigenfunctions and different particular cases are given in the Appendix.

For simplicity we consider only real potentials. In this case $q(x)$ may have either symmetric or antisymmetric boundary conditions at $x\rightarrow\pm\infty$. Without losing generality, one may take $q(+\infty)=|q_0|$ and $q(-\infty)=\pm|q_0|$. Then, using the trial potential $q^0(x)=u(x)$ as in Eq. (3.1) with $|q_1|=0$, $d=x_0$, $\theta_1=0$, and $\theta_0=0$ or π [a symmetric or antisymmetric *black box*; see Eqs. (A4)–(A6)], Eq. (2.6) may be written in as

$$\lambda=\lambda^0+\frac{\lambda^0\nu^0}{(1+2\nu^0x_0)|q_0|}\times[J_1/f_\pm(x_0)+J_2\exp(2\nu^0x_0)-|q_0|/\nu^0], \quad (3.2)$$

where $\nu^0=[|q_0|^2-(\lambda^0)^2]^{1/2}$,

$$J_1=\int_0^{x_0}[q(x)\pm q(-x)]f_\pm(x)dx,$$

$$J_2=\int_{x_0}^{+\infty}[q(x)\pm q(-x)]\exp(-2\nu^0 x)dx,$$

the upper sign $(+)$ and $f_+(x)=\cos(2\lambda^0 x)$ correspond to potentials with a symmetric boundary condition, $q(-\infty)=|q_0|$, and the lower sign $(-)$ and $f_-(x)=\sin(2\lambda^0 x)$ correspond to potentials with an antisymmetric boundary condition $q(-\infty)=-|q_0|$.

A. Gray box

As the first example let us consider the gray box [Eq. (3.1), $q(x)=u(x)$ with $|q_1|\neq 0$, $d=x_1$, and $\theta_0=\theta_1=0$]; see Fig. 1(a). The appropriate spectral problem for such a potential has been solved in [4], where a transcendent equation for the evaluation of the discrete spectrum has been given. The variational method gives the following formula for approximate λ:

$$\lambda=\lambda^0+\frac{\nu^0}{(1+2\nu^0 x_0)|q_0|}\left[\frac{\nu^0}{\lambda^0}|q_1|-\frac{\lambda^0}{\nu^0}(|q_0|-|q_1|)\right.$$
$$\left.\times\{1-\exp[-2\nu^0(x_1-x_0)]\}\right], \tag{3.3}$$

where λ^0 is defined from Eq. (A4). From Eq. (2.10) we have taken the following equation for x_0: $x_0=x_1(1-|q_1/q_0|^2)$. Figure 2(a) shows the dependence of the first three eigenvalues on x_1, evaluated with the help of Eq. (3.3), for $|q_1|=0.2$. The solid curve corresponds to the exact values, found from Eq. (A7) (see [4]). It is clear that in a wide region of x_1 the deviation of approximate values from exact values is less than 10%.

B. tanh pulse

Let us consider the initial pulse with antisymmetric boundaries, for instance, $q(x)=|q_0|\tanh(x/x_1)$ [see Fig. 1(b)] for which the spectral problem has also been solved exactly. In Ref. [5] the following equations for eigenvalues were given:

$$\lambda_1=0,\quad \lambda_{2n}=-\lambda_{2n+1}=|q_0|[1-(1-nx_1/|q_0|)^2]^{1/2},$$
$$n=1,\ldots,N_0,\quad N_0=[|q_0|/x_1]. \tag{3.4}$$

Having found λ^0 from Eq. (A5) and inserting it into Eq. (3.2), one takes the approximation for λ. The condition (2.10) gives the following estimation for optimal values: $x_0=x_1$. The comparison between approximate and exact values of λ is plotted in Fig. 2(b). It shows that the approximation of the third and fourth eigenvalues starts to deteriorate as x_1 increases. As is known, it is usual for variational methods. The reason for such deterioration is that we have not made an optimal choice of the variational parameter x_0. One should calculate x_0 by following the variational scheme instead of using the condition (2.10).

C. Dark Gaussian pulse

As the last example we consider the initial pulse $q(x)=|q_0|[1-\alpha\exp(-x^2/x_1^2)]$, the dark Gaussian pulse [see Fig. 1(c)]. We compared approximate values of λ_n, taken from Eq. (3.2), where λ^0 is evaluated from Eq. (A4), $x_0=(\pi/8)^{1/2}\alpha(2^{3/2}-\alpha)x_1$, with the results of numerical simulations of Eq. (2.1). Our numerical scheme is based on the approximation of the potential with the piecewise constant analog (step on $x\sim 0.01$), for which there is a matrix transformation between the coefficients of eigenfunctions at $-\infty$ and $+\infty$. Making use of the bisection method in the interval $[-|q_0|,|q_0|]$ we find such a λ at which $a(\lambda)\approx 0$ with accuracy $\sim 10^{-5}$. In Fig. 2(c) approximate and numerically calculated values are plotted. As in the previous cases, one can see good agreement between the first eigenvalue and its approximation.

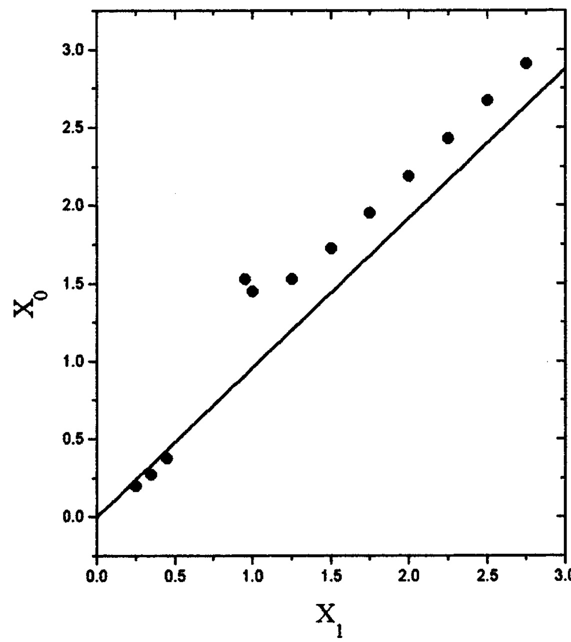

FIG. 3. Comparison between optimal parameter x_0 (circles) and its estimation from the condition (2.10) (solid line) for the gray box, $q_1=0.2$.

Now we estimate the influence of the fact that we evaluate optimal values of the parameter (in our examples x_0) not varying Eq. (2.6), but from Eq. (2.10). For all the potentials considered above in a wide region of parameters (x_1,α) such a choice of x_0 does not deteriorate our estimations. In Fig. 3 the dependence of approximate and optimal values of x_0 on x_1 for the gray box (with $q_1=0.2$) is plotted. Optimal values obtained numerically following the traditional variational scheme from Eq. (3.3). Here we should note that the functional (3.3) may be a monotonic function of x_1, e.g., for $\sim 0.5<x_1<\sim 0.95$, but for some values of x_1 it has several (up to three) extrema and we choose those corresponding to the lowest λ. One can see the good agreement between approximate and optimal values of x_0. Thus our condition for finding the optimal value of the variational parameter is simple and at the same time effective in order to obtain the lowest eigenvalue, but it should be improved to obtain higher-order eigenvalues.

IV. CONCLUSION

We have illustrated the possibility of a variational reformulation of the Zakharov-Shabat eigenvalue problem and the use of it to obtain parameters of dark solitons emerging from the given initial pulse. We have put forth a procedure of finding the discrete spectrum based on the eigenfunctions and eigenvalues of a trial potential. As such a potential one can choose the function (3.1). It should be noted that on choosing a trial potential of the form (3.1) we have the implicit equation for λ^0 [see Eqs. (A4) and (A5)], but trial eigenfunctions are represented by simple trigonometric and exponential functions. The procedure we have utilized in the present paper gives trial functions that satisfy the boundary conditions and the condition of orthogonality. It allows us to determine both the lowest- and higher-order eigenvalues. The fact that Eq. (1.1) is exactly integrable by means of the IST method gives a simple rule for evaluating optimal (or almost optimal) values of variational parameters. Applications of this approach have shown qualitative agreement between exact and approximate eigenvalues of the problem. Of course, in the general case of the linear eigenvalue problem, by using our procedure one has to vary the functional with respect to unknown parameters and solve the variational equations obtained to take optimal values of parameters. As usual, in the variational method, one can obtain increased accuracy by using more variational parameters. For this purpose one can take the trial potential in the multistep form.

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APPENDIX: SCATTERING DATA FOR STEPWISE POTENTIAL (3.1)

Eigenfunctions of the Zakharov-Shabat problem (2.1) and (2.2) with the potential (3.1) have the forms

$$\Psi(x,\lambda)=\begin{cases}\begin{pmatrix}1\\ C_0e^{i\theta_0}\end{pmatrix}\exp(-i\zeta_0 x), & x<-d\\[2ex] A\begin{pmatrix}1\\ C_1e^{i\theta_1}\end{pmatrix}\exp(-i\zeta_1 x)+B\begin{pmatrix}-C_1e^{-i\theta_1}\\ 1\end{pmatrix}\exp(i\zeta_1 x), & |x|\leqslant d\\[2ex] a(\lambda)\begin{pmatrix}1\\ C_0\end{pmatrix}\exp(-i\zeta_0 x)+b(\lambda)\begin{pmatrix}-C_0\\ 1\end{pmatrix}\exp(i\zeta_0 x), & x>d,\end{cases} \tag{A1}$$

where $C_j=i(\lambda-\zeta_j)/|q_j|,\ \zeta_j=(\lambda^2-|q_j|^2)^{1/2},\ j=0,1,$

$$A=[1+C_0C_1e^{i(\theta_0-\theta_1)}]e^{i(\zeta_0-\zeta_1)d}/(1+C_1^2),$$
$$B=(C_0e^{i\theta_0}-C_1e^{i\theta_1})e^{i(\zeta_0+\zeta_1)d}/(1+C_1^2), \tag{A2}$$

and the spectral data are determined from

$$\begin{aligned}a(\lambda)=&e^{2i\zeta_0 d}[e^{-2i\zeta_1 d}(1+C_0C_1e^{i\theta_1})(1+C_0C_1e^{i(\theta_0-\theta_1)})\\ &+e^{2i\zeta_1 d}(C_0-C_1e^{-i\theta_1})\\ &\times(C_0e^{i\theta_0}-C_1e^{i\theta_1})]/[(1+C_0^2)(1+C_1^2)],\\ b(\lambda)=&[e^{-2i\zeta_1 d}(C_1e^{i\theta_1}-C_0)(1+C_0C_1e^{i(\theta_0-\theta_1)})\\ &+e^{2i\zeta_1 d}(C_0e^{i\theta_0}-C_1e^{i\theta_1})\\ &\times(1+C_0C_1e^{-i\theta_1})]/[(1+C_0^2)(1+C_1^2)].\end{aligned} \tag{A3}$$

Let us consider some particular cases, which can be used as trial potentials.

(i) *Black box* ($\theta_0=\theta_1=|q_1|=0$); *see* [1]. The main formula can be easily taken from Eqs. (A1)–(A3). The discrete spectrum is defined from

$$\lambda=\nu\cot(2\lambda d),\quad \nu=(|q_0|^2-\lambda^2)^{1/2}=\zeta_0/i. \tag{A4}$$

The roots of Eq. (A4) lie symmetrically in the region $[-|q_0|,|q_0|]$.

(ii) *Antisymmetric black box* ($\theta_0=\pi,\ \theta_1=|q_1|=0$). The discrete spectrum is defined from

$$\lambda=-\nu\tan(2\lambda d). \tag{A5}$$

The roots lie symmetrically and $\lambda_1=0$. Eigenfunctions for both potentials may be presented as

$$\Psi=\begin{cases}\begin{pmatrix}1\\ \pm C_0\end{pmatrix}\exp(\nu x), & x<-d\\ A\begin{pmatrix}1\\ 0\end{pmatrix}\exp(-i\lambda x)+B\begin{pmatrix}0\\ 1\end{pmatrix}\exp(i\lambda x), & |x|\leqslant d\\ b(\lambda)\begin{pmatrix}-C_0\\ 1\end{pmatrix}\exp(-\nu x), & x>d,\end{cases} \tag{A6}$$

where $A=\exp[-(i\lambda+\nu)d]$ and $B=\pm C_0\exp[(i\lambda-\nu)d]$. In the first case we choose the upper sign (+) in Eq. (A6) and

$$b(\lambda)=i\beta,\quad C_0=i\beta\exp(-2i\lambda d),$$

$$\beta=\mathrm{sgn}\{\nu/[|q_0|\sin(2\lambda d)]\}.$$

In the second case we choose the lower sign (−) in Eq. (A6) and

$$b(\lambda)=-\beta, \quad C_0=\beta \exp(-2i\lambda d),$$

$$\beta=\mathrm{sgn}\{\nu/[|q_0|\cos(2\lambda d)]\}.$$

Note that only positive values of ν must be considered.

(iii) *Gray box* ($\theta_0=\theta_1=0$, $|q_1|\neq 0$). We give here only the equation for discrete spectrum (see [10]):

$$\lambda^2-|q_0 q_1|=\nu\zeta_1\cot(2\zeta_1 d). \tag{A7}$$

The roots of Eq. (A7) satisfy the condition $|q_1|\leqslant|\lambda|\leqslant|q_0|$.

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